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Proof of a Conjecture on the Minimum ABS Index of Bicyclic Graphs

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Introduction

- The atom-bond sum-connectivity (ABS) index of a connected graph $G=(V,E)$ is defined as

$$ABS(G)=\sum_{v_iv_j\in E}\sqrt{\frac{d_i+d_j-2}{d_i+d_j}},$$

where $V=\{v_1,v_2,\ldots,v_n\}$, $d_i=d(v_i)$ is the degree of vertex v_i.
- The predictive applicability of the ABS index is comparable to those of some famous indices, such as the Randić index, sum-connectivity index, and atom-bond connectivity index.
- The characterization of extremal graphs among given graph class attracts attentions.
- Most known extremal results can be found in the survey [1].

Research objectives

Let $m_{i,j}=\left|\{uv\in E\mid d(u)=i,d(v)=j\}\right|$, $1\le i\le j\le n-1$.

Let $\mathcal{B}_n$ be the set of bicyclic graphs of order n, and $\mathcal{B}_n^*=\{G\in\mathcal{B}_n\mid m_{2,2}=n-4,m_{2,3}=4,m_{3,3}=1\}$.

Define $f(x,y)=\sqrt{(x+y-2)/(x+y)}$, $x,y\ge 1$.

- The chemical bicyclic graphs with minimum ABS index were characterized recently in [2].

Lemma 1. If G is a chemical bicyclic graph of order $n\ge 6$, then
$ABS(G)\ge f(3,3)+4f(2,3)+(n-4)f(2,2)$, with equality iff $G\in\mathcal{B}_n^*$.

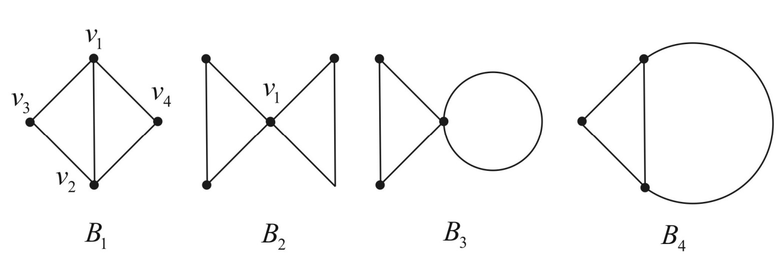
- Lemma 1. was conjectured to be also true for general bicyclic graphs in [3]. We confirm this conjecture. That is, we will prove the following result.

Theorem 1. If G is a (chemical) bicyclic graph of order $n\ge 4$, then
$ABS(G)\ge f(3,3)+4f(2,3)+(n-4)f(2,2)$, with equality iff $G\in\mathcal{B}_n^*$.

Preliminaries

➢ Bicyclic graphs with given degree sequence having minimum ABS index

Let B_1,B_2,B_3,B_4 be the following four bicyclic graphs.



If $\pi=(d_1,d_2,\ldots,d_n)$ is the degree sequence of a bicyclic graph, since $\sum_{i=1}^n d_i=2n+2$, π should be one of the following four cases. Define a special breadth-first searching (BFS) bicyclic graph $B^*(\pi)$ as follows:

Case 1. $d_1\ge d_2\ge 3$ and $d_n=1$. Let $B^*(\pi)$ be a graph obtained from B_1, and the remaining vertices appear in a BFS ordering;

Case 2. $d_1\ge 5, d_2=2$, and $d_n=1$. Let $B^*(\pi)$ be the graph obtained from B_2 by attaching d_1-4 pendent paths of almost equal lengths at vertex v_1;

Case 3. $d_1=4$ and $d_2=d_3=\cdots=d_n=2$. Let $B^*(\pi)=B_3$;

Case 4. $d_1=d_2=3$ and $d_3=d_4=\cdots=d_n=2$. Let $B^*(\pi)=B_4$.

From the Theorem 2.2 in [3], we easily have the following result.

Theorem 2. If $\pi=(d_1,d_2,\ldots,d_n)$ is the degree sequence of a bicyclic graph, then $B^*(\pi)$ has minimum ABS index in $\mathcal{B}_n$.

➢ A graph transformation

Denote by $G(k,l)$ the graph obtained from a non-trivial connected graph G by attaching two pendent paths $uu_1u_2\cdots u_k$ and $uv_1v_2\cdots v_l$ at vertex $u\in V(G)$, $k\ge l\ge 0$. We prove the following result.

Theorem 3. If $k\ge l\ge 0$ and $k\ge 1$, then
$$ABS(G(k+l,0))\le ABS(G(k,l))\le ABS(G(k+l-1,1)),$$
with the two equalities iff $l=0$ and $l=1$, respectively.

Main Result

Let $\Phi(n)=f(3,3)+4f(2,3)+(n-4)f(2,2)\approx n/\sqrt{2}+1.08646$.

Proof of Theorem 1. For $n=4,5$, the conclusion can be easily confirmed, because $|\mathcal{B}_4|=1$ and $|\mathcal{B}_5|=3$.

Suppose G is a bicyclic graph of order $n\geq 6$ with minimum ABS index, and $\pi=(d_1,d_2,\ldots,d_n)$ its non-increasing degree sequence. From Theorem 2, $ABS(G)=ABS(B^*(\pi))$. Recall that π should be one of the following four cases.

Case 1. $d_1\geq d_2\geq 3$ and $d_n=1$. Then $B^*(\pi)$ is obtained from B_1 by attaching a tree T_i at vertex v_i, $i=1,2,3,4$. From Theorem 3, each T_i should be a pendent path, say of length $l_i\geq 0$. Denote $B^*(\pi)$ by $B(l_1,l_2;l_3,l_4)$. By symmetry, assume $l_1\geq l_2$ and $l_3\geq l_4$. Since $ABS(G)=ABS(B^*(\pi))$ is minimum, from the monotonicity of $f(x,y)$, it is easily shown that $ABS(G)=ABS(B(0,0;n-4,0))\approx n/\sqrt{2}+1.10799>\Phi(n)$.

Case 2. $d_1\geq 5, d_2=2$, and $d_n=1$. From Theorem 3, $B^*(\pi)$ is the graph obtained from B_2 by attaching a pendent path of length $n-5$ at v_1. If $n=6$, $ABS(G)\approx 5.61133>\Phi(6)\approx 5.3291$. Otherwise, if $n\geq 7$, then

$$ABS(G)\approx n/\sqrt{2+1.26759}>\Phi(n).$$

Case 3. $d_1=4$ and $d_2=d_3=\cdots=d_n=2$. Then $B^*(\pi)=B_3$, and $ABS(G)\approx n/\sqrt{2}+1.14467>\Phi(n)$.

Case 4. $d_1=d_2=3$ and $d_3=d_4=\cdots=d_n=2$. By counting the edges of G that are incident to vertices of degree $i=2,3$, we have $2m_{2,2}+m_{2,3}=2n-4$ and $m_{2,3}+2m_{3,3}=6$. Moreover, $m_{3,3}=0,1$, because either $v_1v_2\notin E$ or $v_1v_2\in E$. If $m_{3,3}=0$, then $m_{2,2}=n-5$ and $m_{2,3}=6$. Hence $ABS(G)\approx n/\sqrt{2}+1.11205>\Phi(n)$. Otherwise, if $m_{3,3}=1$, then $G\in\mathcal{B}_n^*$ and $ABS(G)=\Phi(n)$.

Combining Cases 1-4, it holds that $ABS(G)=\Phi(n)$, with equality iff $G\in\mathcal{B}_n^*$.

References

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